Algebraic Topology Semestral Examination, 2006, M.Math 1st Year

Attempt any four questions. Each question carries 15 marks. You may consult books and notes.

1. (i): Compute the fundamental group of $S^n \vee \mathbb{RP}(n)$ $(n \geq 2)$

   (ii): Compute $\mathbb{Q}/\mathbb{Z} \otimes \mathbb{Q}/\mathbb{Z}$.

   (iii): Let $\alpha : \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ be any abelian group homomorphism. Construct a continuous map $f : S^1 \to S^1 \vee S^1$ such that $f_* : H_1(S^1, \mathbb{Z}) \to (S^1 \vee S^1, \mathbb{Z})$ is the given homomorphism α.

2. (i): Let $\{C_., \partial^C_.\}$ be the chain complex defined by:
$$\begin{aligned} C_i &= \mathbb{Z}a_i \oplus \mathbb{Z}b_i \quad (-\infty < i < \infty) \\ \partial^C_i a_i &= a_{i-1}; \;\; \partial^C_i b_i = 0 \quad \text{for } i \text{ odd} \\ \partial^C_i a_i &= 0; \;\; \partial^C_i b_i = b_{i-1}; \quad \text{for } i \text{ even.} \end{aligned}$$
   Compute the homologies $H_i(C_.)$ for all $-\infty < i < \infty$.

   (ii): Let $\{D_., \partial^D_.\}$ be the chain complex defined by:
$$\begin{aligned} D_i &= \mathbb{Z}c_i \quad (-\infty < i < \infty) \\ \partial^D_i c_i &= 0 \quad \text{for all } i. \end{aligned}$$
   Show that the map $f_. : C_. \to D_.$ defined by:
$$\begin{aligned} f_i(a_i) &= c_i; \;\; f_i(b_i) = 0 \quad \text{for } i \text{ odd} \\ f_i(a_i) &= 0; \;\; f_i(b_i) = c_i \quad \text{for } i \text{ even} \end{aligned}$$
   is a chain map satisfying $f_i : C_i \to D_i$ is surjective for all i but $f_* : H_i(C_.) \to H_i(D_.)$ is not surjective for any i.

   (iii): Compute $\mathrm{Tor}(\mathbb{Z}_{28}, \mathbb{Z}_{49})$.

3. (i): Express $X = S^1 \vee S^1$ as a simplicial complex, write down the simplicial chain complex of X (explicitly describing the boundary maps), and compute its homology using this chain complex.

   (ii): Show that X above cannot be homotopically equivalent to a compact manifold of any dimension (orientable or non-orientable) (*Hint:* Use Poincare Duality).

   (iii): Find a non-compact 2-manifold to which X is homotopically equivalent.

4. (i): Let A be any abelian group. Show that the group $\hom_{\mathbb{Z}}(A, \mathbb{Z})$ is torsion free.

   (ii): Prove that for any topological space X, the first integer cohomology $H^1(X, \mathbb{Z})$ is always torsion free.

   (iii): Prove that the $(n-1)$-th integer homology $H_{n-1}(M, \mathbb{Z})$ of a compact connected orientable n-manifold M is always torsion free.

5. (i): Let $M = \mathbb{CP}(n)$. Show that $H^{2n}(M, \mathbb{Z}) = \mathbb{Z}$.

   (ii): For M as in (i) above, and a continuous map $f : M \to M$, define the degree of f by the formula:
$$f_*(\mu_M) = (\deg f)\mu_M$$
   where $H_{2n}(M, \mathbb{Z}) = \mathbb{Z}\mu_M$ and $f_* : H_{2n}(M, \mathbb{Z}) \to H_{2n}(M, \mathbb{Z})$ is the map induced on top integer homology. Show that
$$f^* \mu^*_M = (\deg f)\mu^*_M$$
   where μ^*_M (satisfying $\langle \mu^*_M, \mu_M \rangle = 1$) is a generator of $H^{2n}(M, \mathbb{Z})$, and $f^* : H^{2n}(M, \mathbb{Z}) \to H^{2n}(M, \mathbb{Z})$ is the map on top cohomology induced by f.

   (iii): Show that for f as in (ii), M as in (i), $\deg f = k^n$ for some integer k.